On Integrable Solutions of Impulsive Delay Differential Equations

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1 Introduction

The belonging of solutions to a certain function space is a characteristic property for studying the asymptotic behavior of solutions of differential equations. Many works are concerned with the connection between the properties of solutions and stability. We name here the monographs [1-3] on ordinary differential equations and the works [4-9] on functional differential equations. For differential equations with impulses this problem was investigated in [10-12] for ordinary differential equations and in [13] for equations with delay.

The present paper deals with the following problems:

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admissibility of a pair of spaces for a differential operator, i.e. action conditions for this operator in corresponding function spaces;

admissibility of a pair of spaces for a differential equation, i.e. the conditions of belonging of all solutions to a certain space if provided that the right hand side belongs to the other space;

connection between admissibility and exponential stability for impulsive differential equations.

All function spaces considered are the space of locally integrable functions and its subspaces. Explicit conditions for existence of integrable solutions and for exponential stability are obtained as corollaries of these results.

The present paper is organized as follows. In section 2 the equation studied is described and the hypotheses are introduced. Section 3 deals with auxiliary results. In particular the solution representation formula is given and the properties of certain spaces of differentiable on the half-line functions are described. The proofs of these results are presented in the last section 7. In section 4 admissibility of a pair of spaces is considered. In section 5 stability problems are investigated. Finally, section 6 gives explicit stability results.

In conclusion we outline that the present work can be treated as [13] continued. This paper dealt with the same problems in the space of essentially bounded functions.

2 Preliminaries

Let $0 = \tau_0 < \tau_1 < \dots$ be the fixed points, $\lim_{j\to\infty} \tau_j = \infty$, $\mathbf{R}^n$ be the space of n-dimensional column vectors $x = col(x_1, \dots, x_n)$ with the norm $\| x \| = \max_{1\le i\le n} | x_i |$, by the same symbol $\| \cdot \|$ we denote the corresponding matrix norm,

E_n is an $n \times n$ unit matrix,

$\chi_e : [0,\infty) \to \mathbf{R}$ is the characteristic function of the set $e : \chi_e(t) = 1$, if $t \in e$, and $\chi_e(t) = 0$, otherwise.

$\mathbf{L}$ is a space of Lebesgue measurable functions $x : [0,\infty) \to \mathbf{R}^n$ integrable on any finite segment $[t, t+1]$,

$\mathbf{L}_\infty \subset \mathbf{L}$ is a Banach space of essentially bounded functions $x : [0,\infty) \to \mathbf{R}^n$, $\| x \|_{\mathbf{L}_\infty} = vraisup_{t\ge 0} \| x(t) \|$,

$\mathbf{L}_p \subset \mathbf{L}$ $(1 \leq p < \infty)$ is a Banach space of functions $x : [0, \infty) \to \mathbf{R}^n$ such that $\int_0^\infty \parallel x(t) \parallel^p dt < \infty$, with a norm

$$\parallel x \parallel_{\mathbf{L}_p} = \left(\int_0^\infty \parallel x(t) \parallel^p dt \right)^{1/p},$$

$\mathbf{M}_p \subset \mathbf{L}$ is a Banach space of functions $x : [0, \infty) \to \mathbf{R}^n$ such that

$$\mu = \sup_{t>0} \left(\int_t^{t+1} \parallel x(t) \parallel^p dt \right)^{1/p} < \infty, \ 1 \leq p < \infty, \ \ \parallel x \parallel_{\mathbf{M}_p} = \mu.$$

$\mathbf{PAC}(\tau_1, \dots, \tau_j, \dots)$ is a linear space of functions $x : [0, \infty) \to \mathbf{R}^n$ absolutely continuous on each interval $[\tau_j, \tau_{j+1})$, with jumps at the points τ_j. We assume that functions in $\mathbf{PAC}$ are right continuous.

The same function spaces will be considered for intervals different from $[0, \infty)$ if it does not lead to misunderstanding.

For spaces of matrix valued functions we use the same notation as for vector valued functions.

We consider a delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t) x[h_k(t)] = f(t), \ t \geq 0, \ x(t) \in \mathbf{R}^n, \tag{1}$$

$$x(\xi) = \varphi(\xi), \xi < 0, \tag{2}$$

with impulsive conditions

$$x(\tau_j) = B_j x(\tau_j - 0) + \alpha_j, \ j = 1, 2, \dots, \tag{3}$$

under the following assumptions:

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ are fixed points, $\lim_{j \to \infty} \tau_j = \infty$;

(a2) $f \in \mathbf{L}, \ A_k \in \mathbf{L}, \ k = 1, 2, \dots, m$;

(a3) $h_k : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions,

$$h_k(t) \leq t, \ k = 1, \dots, m;$$

(a4) $\varphi : (-\infty, 0) \to \mathbf{R}^n$ is a Borel measurable bounded function;

(a5) $B_j \in \mathbf{R}^{n \times n}, \ B = \sup_j \parallel B_j \parallel < \infty$;

(a6) $K = \sup_{t,s>0} \left\{ \frac{i(t,s)}{t-s}, \ i(t,s) \neq 1 \right\} < \infty$.

Here $i(t,s)$ is a number of points τ_j belonging to the interval (s,t).
We denote $b = \max\{B, 1\},\ I = max\{K, 1\}$.

Remark. One can easily see that (a6) is satisfied if $\tau_{j+1} - \tau_j \geq \rho > 0$.
Definition . A function $x \in \mathbf{PAC}$ is said to be **a solution of the impulsive equation (1),(2),(3)** with the initial function $\varphi(t)$ if (1) is satisfied for almost all $t \in [0, \infty)$ and the equalities (3) hold.

Below we use a linear differential operator

$$(\mathcal{L}x)(t) = \dot{x}(t) + \sum_{k=1}^{m} A_k(t)x[h_k(t)],\ x(\xi) = 0,\ \xi < 0. \tag{4}$$

3 Auxiliary results

In [13] the solution representation formula for (1)-(3) is presented if provided that more restrictive conditions than (a1)-(a6) hold. Precisely, instead of (a2) it was assumed that f and A_k are in $\mathbf{L}_\infty$. However the proof of this formula preserves in the more general case $f, A_k \in \mathbf{L}$. Thus the following result is valid.

Lemma 1 [13] *Suppose the hypotheses (a1)-(a6) hold.*

Then there exists one and only one solution of the equation (1) -(3) satisfying $x(0) = \alpha_0$ and it can be presented as

$$x(t) = \int_0^t X(t,s)f(s)ds - \sum_{k=1}^{m} \int_0^t X(t,s)A_k(s)\varphi[h_k(s)]ds + \sum_{0 \leq \tau_j \leq t} X(t,\tau_j)\alpha_j, \tag{5}$$

with $\varphi(\zeta) = 0,\ if\ \zeta \geq 0$.

The matrix $X(t,s)$ in (5) for a fixed s as a function of t is a solution of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x[h_k(t)] = 0,\ t \geq s,\ x(t) \in \mathbf{R}^{n\times n},$$

$$x(\xi) = 0,\ \xi < s,\ x(s) = E_n;\ x(\tau_j) = B_j x(\tau_j - 0),\ \tau_j > s.$$

We assume $X(t,s) = 0,\ t < s$.

Definition. The matrix $X(t,s)$ is said to be **a fundamental matrix**, $X(t,0)$ is said to be **a fundamental solution**. An operator

$$(Cf)(t) = \int_0^t X(t,s)f(s)ds$$

is said to be **a Cauchy operator** of the equation (1)-(3).

For studying the equation (1)- (3) we introduce an auxiliary equation

$$(\mathcal{L}_0 x)(t) \equiv \dot{x}(t) + ax(t) = z(t),\ t \geq 0,\ x(t) \in \mathbf{R}^n, \tag{6}$$

$$x(\tau_j) = B_j x(\tau_j - 0). \tag{7}$$

By

$$(C_0 z)(t) = \int_0^t X_0(t,s)z(s)ds$$

the Cauchy operator of the equation (6),(7) is denoted.

Lemma 2 [13] *Suppose the hypotheses (a5) and (a6) hold and* $\nu = a - I\ln b > 0$.

Then

$$\| X_0(t,s) \| \leq e^{-\nu(t-s)}.$$

For each space $\mathbf{L}_p$ we construct a subspace of **PAC** as follows. Denote by $\mathbf{D}_p$ a linear space of functions $x \in$ **PAC** satisfying (7) and such that $x \in \mathbf{L}_p,\ \dot{x} \in \mathbf{L}_p$. This space is normed, with a norm

$$\| x \|_{\mathbf{D}_p} = \| x \|_{\mathbf{L}_p} + \| \dot{x} \|_{\mathbf{L}_p}.$$

Lemma 3 *Suppose the hypotheses (a5) and (a6) hold.*

Then $\mathbf{D}_p,\ 1 \leq p < \infty$, *is a Banach space.*

The proof is presented in section 7.

Remark. Lemma 3 remains valid if $\mathbf{L}_p$ is changed by a Banach space $\mathbf{B} \subset \mathbf{L}$ if provided that the topology in $\mathbf{B}$ is stronger than the topology in $\mathbf{L}$. In particular $\mathbf{B} = \mathbf{L}_\infty$ or $\mathbf{B} = \mathbf{M}_p$ are suitable.

The following assertion supplements Lemma 3.

Lemma 4 *Suppose the hypotheses (a5) and (a6) hold and* $a - I\ln b > 0$.

Then the set $\tilde{\mathbf{D}}_p = \{x \in \mathbf{PAC} \mid \ \dot{x} + ax \in \mathbf{L}_p,\ x(\tau_j) = B_j x(\tau_j - 0)\}$ *coincides with* $\mathbf{D}_p$*, and the norm*

$$\| x \|_{\tilde{\mathbf{D}_p}} = \| x(0) \| + \| \dot{x} + ax \|_{\mathbf{L}_p} \tag{8}$$

is equivalent to the norm $\| \cdot \|_{\mathbf{D}_p}$.

The proof is also in section 7.

4 Admissibility of pairs

Definition. The pair $(\mathbf{D}_p, \mathbf{L}_p)$ is said to be **admissible for a differential operator** $\mathcal{L} : \mathbf{PAC} \to \mathbf{L}$ if $\mathcal{L}(\mathbf{D}_p) \subset \mathbf{L}_p$.

Definition. Suppose the initial function φ satisfies the hypothesis (a4) and it is fixed. The pair $(\mathbf{L}_p, \mathbf{D}_p)$ is said to be **admissible** for the equation (1)-(3) if for any $f \in \mathbf{L}_p,\ \alpha_j \in \mathbf{R}^n$ the solution is in $\mathbf{D}_p$.

The pair $(\mathbf{L}_p, \mathbf{D}_p)$ is said to be **admissible on the whole** for the equation (1)-(3) if for any $f \in \mathbf{L}_p,\ \alpha_j \in \mathbf{R}^n$ and any initial function φ satisfying (a4) the solution is in $\mathbf{D}_p$.

Remarks. 1. For ordinary differential equations the admissibility of the pair $(\mathbf{L}_p, \mathbf{L}_\infty)$ is usually considered. However this admissibility is the consequence of the admissibility of pair $(\mathbf{L}_p, \mathbf{D}_p)$. In fact if $x \in \mathbf{D}_p$ then for $a \in \mathbf{R}$ $\ \dot{x} + ax \in \mathbf{L}_p$. Let $a - I \ln\ b > 0$. Then by Lemma 2 $x \in \mathbf{L}_\infty$, therefore the pair $(\mathbf{L}_p, \mathbf{L}_\infty)$ is admissible for the differential equation. Besides this, under our approach admissibility of the pair $(\mathbf{L}_p, \mathbf{D}_p)$ is treated more naturally than of the pair $(\mathbf{L}_p, \mathbf{L}_\infty)$.

2. It is to be noted that the recent monograph of C.Corduneanu [9] deals with admissibility of pairs of spaces for integrodifferential equations (and for general functional differential equations as well).

Consider operators

$$(Hx)(t) = \sum_{k=1}^{m} A_k(t)x[h_k(t)];\ x(\xi) = 0,\ \xi < 0, \tag{9}$$

$$(\mathcal{L}x)(t) = \dot{x}(t) + (Hx)(t).$$

Under the hypotheses (a1)-(a3), (a5)-(a6) H acts from **PAC** to **L**.

Theorem 1 *Suppose the hypotheses (a1)-(a3), (a5),(a6) hold and there exists* $\nu > 0$ *such that* $A_k^\nu \in \mathbf{M}_p$, *where*

$$A_k^\nu(t) = e^{\nu[t-h_k(t)]}A_k(t),\ 1 \le p < \infty.$$

Then operators H *and* $\mathcal{L}$ *act from* $\mathbf{D}_p$ *to* $\mathbf{L}_p$ *and they are bounded.*

Proof. Let $a = \nu + I \ln b$ and $x \in \mathbf{D}_p$. Then $z = \dot{x} + ax \in \mathbf{L}_p$ and x can be presented as

$$x(t) = X_0(t,0)x(0) + \int_0^t X_0(t,s)z(s)ds.$$

In sequel $y(h(t)) = 0$, if $h(t) < 0$, and $a^+ = \max\{a, 0\}$.

Thus we obtain

$$(Hx)(t) =$$

$$= \sum_{k=1}^m A_k(t)X_0(h_k(t),0)x(0) + \sum_{k=1}^m \int_0^{h_k^+(t)} A_k(t)X_0(h_k(t),s)z(s)ds. \qquad (10)$$

First we will obtain that a matrix valued function

$$F(t) = \sum_{k=1}^m A_k(t)X_0(h_k(t),0)$$

is in $\mathbf{L}_p$. To this end by Lemma 2

$$\| A_k(t)X_0(h_k(t),0) \| \le \| A_k(t) \| e^{-\nu h_k(t)} =$$

$$= \| A_k(t)e^{\nu(t-h_k(t))} \| e^{-\nu t} = \| A_k^\nu(t) \| e^{-\nu t}.$$

Therefore

$$\int_0^\infty \| A_k^\nu(t) \|^p e^{-\nu p t}dt \le \sup_{n\ge 0} \int_n^{n+1} \| A_k^\nu(t) \|^p dt \sum_{n=0}^\infty e^{-\nu p n} \le$$

$$\le \frac{\| A_k^\nu \|^p_{\mathbf{M}_p}}{1 - e^{-\nu p}}.$$

Hence $F \in \mathbf{L}_p$.

Denote

$$(Pz)(t) = \sum_{k=1}^m \int_0^{h_k^+(t)} A_k(t)X_0(h_k(t),s)z(s)ds.$$

We will prove that P acts in $\mathbf{L}_p$ and it is bounded. To this end

$$\| (Pz)(t) \| \leq \sum_{k=1}^{m} \int_0^{h_k^+(t)} \| A_k(t) e^{\nu[t-h_k(t)]} \| \ e^{-\nu(t-s)} \| z(s) \| ds =$$

$$= \sum_{k=1}^{m} \int_0^{h_k^+(t)} \| A_k^{\nu}(t) \| \ e^{-\nu(t-s)} \| z(s) \| \ ds.$$

Let $p = 1$. Then

$$\| Pz \|_{\mathbf{L}_1} \leq \sum_{k=1}^{m} \int_0^{\infty} \int_0^{t} \| A_k^{\nu}(t) \| \ e^{-\nu(t-s)} \| z(s) \| \ ds \ dt =$$

$$= \sum_{k=1}^{m} \int_0^{\infty} \left(\int_s^{\infty} \| A_k^{\nu}(t) \| \ e^{-\nu(t-s)} dt \right) \| z(s) \| \ ds.$$

Since

$$\int_s^{\infty} \| A_k^{\nu}(t) \| \ e^{-\nu(t-s)} dt \leq \sum_{n=[s]}^{\infty} \int_n^{n+1} \| A_k^{\nu}(t) \| \ e^{-\nu(t-s)} dt \leq$$

$$\leq e^{\nu s} \sum_{n=[s]}^{\infty} e^{-\nu n} \int_n^{n+1} \| A_k^{\nu}(t) \| \ dt \leq e^{\nu s} \| A_k^{\nu} \|_{\mathbf{M}_1} \sum_{n=[s]}^{\infty} e^{-\nu n} \leq$$

$$\leq \| A_k^{\nu} \|_{\mathbf{M}_1} \frac{e^{\nu}}{1 - e^{-\nu}},$$

then

$$\| Pz \|_{\mathbf{L}_1} \leq \frac{e^{\nu}}{1 - e^{-\nu}} \sum_{k=1}^{m} \| A_k^{\nu} \|_{\mathbf{M}_1} \| z \|_{\mathbf{L}_1} .$$

Here $[s]$ is the greatest integer not exceeding s.

Let $1 < p < \infty$. Then similarly we obtain

$$\| Pz \|_{\mathbf{L}_p} \leq \sum_{k=1}^{m} \left[\int_0^{\infty} \| A_k^{\nu}(t) \|^p \left(\int_0^{t} e^{-\nu(t-s)} \| z(s) \| \ ds \right)^p dt \right]^{1/p} =$$

$$= \sum_{k=1}^{m} \left[\int_0^{\infty} \| A_k^{\nu}(t) \|^p \left(\int_0^{t} e^{-\nu(t-s)/2} e^{-\nu(t-s)/2} \| z(s) \| \ ds \right)^p dt \right]^{1/p} \leq$$

$$\leq \sum_{k=1}^{m}\left[\int_0^{\infty} \parallel A_k^{\nu}(t) \parallel^p \left(\int_0^{\infty} e^{-\nu q(t-s)/2} ds\right)^{p/q} \times\right.$$

$$\left.\times \left(\int_0^{t} e^{-\nu p(t-s)/2} \parallel z(s) \parallel^p \ ds\right) \ dt\right]^{1/p} \leq$$

$$\leq \left(\frac{2}{\nu q}\right)^{1/q} \sum_{k=1}^{m}\left[\int_0^{\infty}\int_s^{\infty} \parallel A_k^{\nu}(t) \parallel^p e^{-\nu p(t-s)/2} \parallel z(s) \parallel^p dt\ ds\right]^{1/p},$$

where $q = p/(p-1)$.

By repeating the previous argument we obtain

$$\parallel Pz \parallel_{\mathbf{L}_p} \leq \left(\frac{2}{\nu q}\right)^{1/q} \frac{e^{\nu/2}}{(1-e^{-\nu p/2})^{1/p}} \sum_{k=1}^{m} \parallel A_k^{\nu} \parallel_{\mathbf{M}_p} \parallel z \parallel_{\mathbf{L}_p} .$$

Therefore $Pz \in \mathbf{L}_p$ and operator $P : \mathbf{L}_p \to \mathbf{L}_p$ is bounded.

Operator H defined by (9) in view of (10) can be presented as

$$(Hx)(t) = F(t)x(0) + (Pz)(t), \quad \text{where } z = \dot{x} + ax.$$

Since

$$\parallel Hx \parallel_{\mathbf{L}_p} \leq \parallel F \parallel_{\mathbf{L}_p} \parallel x(0) \parallel + \parallel P \parallel_{\mathbf{L}_p \to \mathbf{L}_p} \parallel \dot{x} + ax \parallel_{\mathbf{L}_p} \leq$$

$$\leq \max\{\parallel F \parallel_{\mathbf{L}_p}, \parallel P \parallel_{\mathbf{L}_p \to \mathbf{L}_p}\} \parallel x \parallel_{\tilde{\mathbf{D}_p}},$$

then by Lemma 4 H acts from $\mathbf{D}_p$ to $\mathbf{L}_p$ and it is bounded. One can easily see that the admissibility of the pair $(\mathbf{D}_p, \mathbf{L}_p)$ for the operator $\mathcal{L}$ is equivalent to admissibility of this pair for H. The proof of the theorem is complete.

Corollary. Suppose the hypotheses (a1)-(a3), (a5),(a6) hold, $A_k \in \mathbf{M}_p,\ 1 \leq p < \infty$ and there exists $\delta > 0$ such that $t - h_k(t) < \delta,\ k = 1, \ldots, m$.

Then H acts from $\mathbf{D}_p$ to $\mathbf{L}_p$ and it is bounded.

Now we proceed to $(\mathbf{L}_p, \mathbf{D}_p)$ admissibility conditions for the problem (1) - (3). To this end consider an auxiliary equation of the type (1), (2)

$$\dot{x}(t) + \sum_{k=1}^{r} H_k(t)x[g_k(t)] = f(t),\ t \geq 0,\ x(t) \in \mathbf{R}^n,$$

$$x(\xi) = \varphi(\xi),\ if \quad \xi < 0. \tag{11}$$

The equation (11) determines a differential operator $\mathcal{M}$

$$(\mathcal{M}x)(t) = \dot{x}(t) + \sum_{k=1}^{r} H_k(t)x[g_k(t)], \ x(\xi) = 0, \ \xi < 0. \qquad (12)$$

Suppose for this equation the hypotheses (a1)-(a4) hold. By $C_{\mathcal{M}}$ we denote the Cauchy operator of this equation.

Lemma 5 *Suppose that for the operators $\mathcal{L}$ and $\mathcal{M}$ defined by (4) and (12) the following conditions are satisfied.*

1. The operators $\mathcal{L}$ and $\mathcal{M}$ act from $\mathbf{D}_p$ to $\mathbf{L}_p$ and they are bounded.

2. $R(\mathcal{M}) = \mathbf{L}_p$, where $R(\mathcal{M})$ is a range of values of the operator $\mathcal{M}$: $\mathbf{D}_p \to \mathbf{L}_p$.

3. The operator $\mathcal{L}C_{\mathcal{M}} : \mathbf{L}_p \to \mathbf{L}_p$ is invertible.

Then $R(\mathcal{L}) = \mathbf{L}_p$ and C acts from $\mathbf{L}_p$ to $\mathbf{D}_p$ and it is bounded.

Proof. Consider an initial value problem

$$\mathcal{L}x = f, \ x(0) = 0, \ x(\tau_j) = B_j x(\tau_j - 0),$$

where $f \in \mathbf{L}_p$ is an arbitrary function. Then $x = C_{\mathcal{M}}(\mathcal{L}C_{\mathcal{M}})^{-1}f$ is the solution of this problem. Therefore $x \in \mathbf{D}_p$, hence $R(\mathcal{L}) = \mathbf{L}_p$.

Let $\mathbf{D}_p^0 = \{x \in \mathbf{D}_p : x(0) = 0\}$. Then by the Banach theorem on an inverse operator the operator $C : \mathbf{L}_p \to \mathbf{D}_p^0$ is bounded. So the operator $C : \mathbf{L}_p \to \mathbf{D}_p$ is also bounded.

Denote

$$\varphi^h(t) = \begin{cases} \varphi[h(t)], & h(t) < 0, \\ 0, & h(t) \geq 0, \end{cases} \quad g(t) = \sum_{k=1}^{m} A_k(t)\varphi^{h_k}(t). \qquad (13)$$

Theorem 2 *Suppose the operators $\mathcal{L}$ and $\mathcal{M}$ defined by (4) and (12) satisfy the conditions of Lemma 5.*

If the function g defined by (13) is in $\mathbf{L}_p$ then pair $(\mathbf{L}_p, \mathbf{D}_p)$ is admissible for the equation (1)-(3).

If there exists $\delta > 0$ such that $t - h_k(t) < \delta$ and the restriction of A_k to $[0, \delta]$ belongs to $\mathbf{L}_p[0, \delta]$, $k = 1, \ldots, m$, then the pair $(\mathbf{L}_p, \mathbf{D}_p)$ is admissible on the whole for the equation (1)-(3).

Proof. Let $f \in \mathbf{L}_p$ and C be the Cauchy operator of (1)-(3). By Lemma 1 solution x of (1)-(3) can be presented as

$$x(t) = (Cf)(t) - (Cg)(t) + \sum_{0 \le \tau_j \le t} X(t, \tau_j)\alpha_j. \tag{14}$$

By Lemma 5 $Cf \in \mathbf{D}_p,\ Cg \in \mathbf{D}_p$. Now we will establish $X(\cdot, \tau_j) \in \mathbf{D}_p,\ j = 1, 2, \dots$. To this end denote

$$Y_j(t) = X(t, \tau_j) - X_0(t, \tau_j),$$

where $X_0(t, s)$ is the fundamental matrix of (6),(7) and $a - I \ln b > 0$.

$$\text{Let} \quad f_j(t) = -\mathcal{L}(X_0(\cdot, \tau_j))(t).$$

Then Y_j is a solution of the problem

$$\mathcal{L}y = f_j,\ t \ge \tau_j,\ y(t) \in \mathbf{R}^{n \times n},$$

$$y(\tau_j) = 0,\ y(\tau_i) = B_i y(\tau_i - 0), \quad i = j + 1, \dots. \tag{15}$$

By Lemma 1 the solution of (15) can be presented as

$$Y_j(t) = (Cf_j)(t),$$

hence

$$X(t, \tau_j) = X_0(t, \tau_j) + (Cf_j)(t). \tag{16}$$

By Lemma 2 $X_0(\cdot, \tau_j) \in \mathbf{D}_p$. Since by the hypothesis of the theorem pair $(\mathbf{D}_p, \mathbf{L}_p)$ is admissible for the operator $\mathcal{L}$ then $f_j \in \mathbf{L}_p$. Therefore by Lemma 5 $Cf_j \in \mathbf{D}_p$. Thus (16) implies $X(\cdot, \tau_j) \in \mathbf{D}_p$ and (14) gives that a solution of (1)- (3) is in $\mathbf{D}_p$. Admissibility of the pair $(\mathbf{L}_p, \mathbf{D}_p)$ for the equation (1)-(3) is proven.

Suppose $t - h_k(t) < \delta$. As g is defined by (13) then $g(t) = 0$ if $t > \delta$. Since for $t \in [0, \delta]$ $A_k \in \mathbf{L}_p[0, \delta]$ and $\varphi^{h_k} \in \mathbf{L}_\infty[0, \delta]$, then $g \in \mathbf{L}_p[0, \delta]$. Therefore for $t \in [0, \infty)$ $g \in \mathbf{L}_p[0, \infty)$. Thus according to the above results the pair $(\mathbf{L}_p, \mathbf{D}_p)$ is admissible on the whole for (1)-(3). The proof of the theorem is complete.

5 Admissibility and stability

This paper deals with exponential stability only. Other types of stability and their connection with properties of the fundamental matrix are presented in [14].

<u>*Definition.*</u> The equation (1)-(3) is said to be **exponentially stable** if there exist positive constants N and λ such that for any initial function $\varphi, f = 0$ and $\alpha_1 = \alpha_2 = \ldots = 0$ for a solution x of (1)-(3) the inequality

$$\| x(t) \| \leq N e^{-\lambda t} \left(\sup_{t<0} \| \varphi(t) \| + \| x(0) \| \right)$$

holds.

Thus the representation (5) yields the following assertion (see [14]).

Theorem 3 *Suppose (a1)-(a6) hold and there exist positive constants N and λ such that the fundamental matrix $X(t,s)$ satisfies the inequality*

$$\| X(t,s) \| \leq N e^{-\lambda(t-s)}, \ t \geq s > 0, \tag{17}$$

and there exists $\delta > 0$ such that $t - h_k(t) < \delta, \ k = 1, \ldots, m$.

Then equation (1)-(3) is exponentially stable.

The following theorem is a main result of this work. It connects admissibility of the pair $(\mathbf{L}_p, \mathbf{D}_p)$ with stability of (1)-(3).

Theorem 4 *Suppose for (1)-(3) the hypotheses (a1)-(a6), hold, $A_k \in \mathbf{M}_p$, $1 \leq p < \infty$, there exists $\delta > 0$ such that $t - h_k(t) < \delta, \ k = 1, \ldots, m$ and for the initial function $\varphi \equiv 0$ the pair $(\mathbf{L}_p, \mathbf{D}_p)$ is admissible for this equation.*

Then the equation (1)-(3) is exponentially stable.

Proof. By Theorem 3 it is sufficient to prove that the estimate (17) exists. In view of Lemma 1 the fundamental matrix $X(t,s)$ as a function of t for a fixed s is a solution of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t) x[h_k(t)] = 0, \ t \geq s, \ x(t) \in \mathbf{R}^{n \times n}, \ x(s) = E_n,$$

$$x(\xi) = 0, \ \xi < s, \ x(\tau_j) = B_j x(\tau_j - 0), \ \tau_j > s. \tag{18}$$

Denote

$$Y(t,s) = e^{\lambda(t-s)} X(t,s), \tag{19}$$

where $\lambda > 0$ is a certain number. Thus

$$Y(s,s) = X(s,s) = E_n \text{ and, besides, } Y(\tau_j, s) = B_j Y(\tau_j - 0, s),\ \tau_j > s.$$

Denote

$$\mathcal{L}_s x = \dot{x}(t) + \sum_{k=1}^{m} A_k(t) x[h_k(t)], t \geq s,\ x(t) \in \mathbf{R}^{n \times n};$$

$$x(\xi) = 0,\ \xi < s,.$$

By substituting $x(t) = y(t) e^{-\lambda(t-s)}$ we obtain

$$(\mathcal{L}_s x)(t) = e^{-\lambda(t-s)} \dot{y}(s) - e^{-\lambda(t-s)} \lambda y(t) + \sum_{k=1}^{m} e^{-\lambda[h_k(t)-s]} A_k(t) y[h_k(t)] =$$

$$= e^{-\lambda(t-s)} \left\{ \dot{y}(t) + \sum_{k=1}^{m} A_k(t) y[h_k(t)] + \right.$$

$$\left. + \sum_{k=1}^{m} e^{\lambda[t-h_k(t)]} A_k(t) y[h_k(t)] - \sum_{k=1}^{m} A_k(t) y[h_k(t)] - \lambda y(t) \right\} =$$

$$= e^{-\lambda(t-s)} \left\{ (\mathcal{L}_s y)(t) - \lambda y(t) + \sum_{k=1}^{m} \left[e^{\lambda(t-h_k(t))} - 1 \right] A_k(t) y[h_k(t)] \right\}.$$

Denote

$$(\mathcal{T}_s y)(t) = \sum_{k=1}^{m} \left[e^{\lambda(t-h_k(t))} - 1 \right] A_k(t) y[h_k(t)] - \lambda y(t),\ t \geq s,$$

$$(\mathcal{M}_s y)(t) = (\mathcal{L}_s y)(t) + (\mathcal{T}_s y)(t).$$

Then

$$(\mathcal{L}_s x)(t) = e^{-\lambda(t-s)} (\mathcal{M}_s y)(t)$$

and $Y(t,s)$ is a fundamental matrix of the problem $\mathcal{M}_0 y = 0,\ y(\tau_j) = B_j y(\tau_j - 0)$.

The corollary of Theorem 1 gives that the operator $\mathcal{L}_s$ acts from $\mathbf{D}_p[s,\infty)$ to $\mathbf{L}_p[s,\infty)$ and it is bounded. By the hypothesis of the theorem a solution of

$\mathcal{L}_s x = f$ together with its derivative is in $\mathbf{L}_p[s,\infty)$ if provided $f \in \mathbf{L}_p[s,\infty)$. Therefore the Cauchy operator C_s of this equation acts from $\mathbf{L}_p[s,\infty)$ to $\mathbf{D}_p[s,\infty)$.

Denote $\mathbf{D}_p^0[s,\infty) = \{x \in \mathbf{D}_p[s,\infty) \mid x(s) = 0\}$. By the hypotheses of the theorem the operator $\mathcal{L}_s : \mathbf{D}_p^0[s,\infty) \to \mathbf{L}_p[s,\infty)$ is bounded. By Lemma 3 the space $\mathbf{D}_p[s,\infty)$ is Banach, therefore its closed subspace $\mathbf{D}_p^0[s,\infty)$ is also Banach. Thus by the Banach theorem on an inverse operator the operator $C_s : \mathbf{L}_p[s,\infty) \to \mathbf{D}_p^0[s,\infty)$ and, consequently, $C_s : \mathbf{L}_p[s,\infty) \to \mathbf{D}_p[s,\infty)$ is bounded.

By Theorem 1 H_s^k acts from $\mathbf{D}_p[s,\infty)$ to $\mathbf{L}_p[s,\infty)$, where $(H_s^k x)(t) = A_k(t)x(h_k(t)),\ x(\xi) = 0,\ \xi < s$. From the assumption $t - h_k(t) < \delta$ we obtain an estimate

$$\| \mathcal{T}_s \|_{\mathbf{D}_p[s,\infty)\to\mathbf{L}_p[s,\infty)} \leq \left(e^{\lambda\delta} - 1\right) \sum_{k=1}^{m} \| H_s^k \|_{\mathbf{D}_p[s,\infty)\to\mathbf{L}_p[s,\infty)} + \lambda.$$

The operator $\mathcal{M}_s C_s = E + \mathcal{T}_s C_s$, with E being an identity operator, has a bounded inverse operator in $\mathbf{L}_p[s,\infty)$ if

$$\| \mathcal{T}_s C_s \|_{\mathbf{L}_p[s,\infty)\to\mathbf{L}_p[s,\infty)} < 1. \tag{20}$$

We prove that for λ being small enough (20) holds. To this end

$$\| \mathcal{T}_s C_s \|_{\mathbf{L}_p\to\mathbf{L}_p} \leq \| \mathcal{T}_s \|_{\mathbf{D}_p\to\mathbf{L}_p} \| C_s \|_{\mathbf{L}_p\to\mathbf{D}_p} \leq \left[(e^{\lambda\delta} - 1) \sum_{k=1}^{m} \| H_s^k \| + \lambda \right] \| C_s \| .$$

Therefore for λ being small enough (20) holds, where λ is obviously independent of s since $\| H_s^k \| \ \leq\ \| H_0^k \|, \| \ C_s \ \| \ \leq \ \| \ C \ \|$.

Operators $\mathcal{L}_s$ and $\mathcal{T}_s$ act continuously from $\mathbf{D}_p[s,\infty)$ to $\mathbf{L}_p[s,\infty)$. Hence the operator $\mathcal{M}_s = \mathcal{L}_s + \mathcal{T}_s$ also possesses this property. Thus by Lemma 5 the Cauchy operator $C_{\mathcal{M}}^s$ of the equation $\mathcal{M}_s y = f$ continuously acts from $\mathbf{L}_p[s,\infty)$ to $\mathbf{D}_p[s,\infty)$.

Similar to (16) we obtain

$$Y(t,s) = X_0(t,s) + (C_{\mathcal{M}}^s f_s)(t). \tag{21}$$

Here $f_s(t) = -\mathcal{M}_s(X_0(\cdot,s))(t),\ a - I\ln b > 0$.

Lemma 2 implies $X_0(\cdot,s) \in \mathbf{D}_p[s,\infty)$. Moreover, this lemma gives the uniform estimate $\| f_s \|_{\mathbf{L}_p[s,\infty)} \leq K$, with K not depending on s.

Therefore we obtain estimates independent of s

$$\| C^s_{\mathcal{M}} f_s \|_{\mathbf{D}_p[s,\infty)} \leq K \| C_{\mathcal{M}} \|,$$

$$\| C^s_{\mathcal{M}} f_s \|_{\mathbf{L}_p[s,\infty)} \leq K \| C_{\mathcal{M}} \| .$$

and

$$\| \frac{d}{dt} C^s_{\mathcal{M}} f_s \|_{\mathbf{L}_p[s,\infty)} \leq K \| C_{\mathcal{M}} \| .$$

Denote $z_s = C^s_{\mathcal{M}} f_s$. Since $z_s(s) = 0$, then $z_s = C^s_0(\dot{z}_s + a z_s)$. By Lemma 2 $C^s_0 : \mathbf{L}_p[s,\infty) \to \mathbf{L}_\infty[s,\infty)$ is bounded, therefore

$$\| C^s_{\mathcal{M}} f_s \|_{\mathbf{L}_\infty[s,\infty)} = \| z_s \|_{\mathbf{L}_\infty[s,\infty)} \leq$$

$$\leq \| C_0 \|_{\mathbf{L}_p \to \mathbf{L}_\infty} \left(\| \dot{z}_s \|_{\mathbf{L}_p[s,\infty)} + a \| z_s \|_{\mathbf{L}_p[s,\infty)} \right) \leq$$

$$\leq (1+a) K \| C_0 \| \quad \| C_{\mathcal{M}} \| .$$

Hence the estimate of the norm of $C^s_{\mathcal{M}} f_s$ in $\mathbf{L}_\infty[s,\infty)$ does not depend on s.

By Lemma 2 and (21) there exists $N > 0$ such that

$$vraisup_{t,s>0} \| Y(t,s) \| \leq N < \infty.$$

Thus (19) implies the exponential estimate (17) for the fundamental matrix of (1)-(3). The proof of the theorem is complete.

6 Explicit stability results

We apply Theorems 2 and 4 to obtaining explicit conditions of exponential stability and of existence of integrable solutions. To this end we prove an auxiliary result.

Lemma 6 *Suppose there exist* $\sigma > 0$ *and* $\rho > 0$ *such that* $\rho \leq \tau_{j+1} - \tau_j \leq \sigma$, $\| B_j \| \leq B < 1$.

Then for the fundamental matrix X_1 *of the equation*

$$\dot{x}(t) = f(t), \; x(\tau_j) = B_j x(\tau_j - 0) \tag{22}$$

the inequality

$$\| X_1(t,s) \| \leq e^{-\eta(t-s-\sigma)} \tag{23}$$

holds, where $\eta = -\frac{1}{\sigma} \ln B$.

Proof. Under the hypotheses of the lemma (see [13])

$$\| X_1(t,s) \| \leq \begin{cases} e^{-\eta(t-s)} & , t-s > \sigma, \\ 1 & , 0 < t-s \leq \sigma. \end{cases}$$

This immediately yields (23).

Denote

$$A_k^{\eta}(t) = A_k(t) e^{\eta(t-h_k(t))}.$$

Theorem 5 *Suppose for the equation (1)-(3) the hypotheses (a3),(a4) and*
(b1) $f \in \mathbf{L}_1,\ A_k^{\eta} \in \mathbf{M}_1$;
(b2) $0 < \rho \leq \tau_{j+1} - \tau_j \leq \sigma$;
(b3) $\| B_j \| \leq B < 1$;
(b4) $g \in \mathbf{L}_1$, *where* g *is defined by (13);*
(b5) $e^{\eta(\sigma+1)} \sum_{k=1}^{m} \| A_k^{\eta} \|_{\mathbf{M}_1} \leq 1 - e^{-\eta}$, *where* $\eta = -\frac{1}{\sigma} \ln B$,
hold.
Then for any solution x *of (1)-(3)* $x \in \mathbf{L}_1,\ \dot{x} \in \mathbf{L}_1$.

Theorem 6 *Suppose for the equation (1)-(3) the hypotheses (a3),(a4) and*
(c1) $f \in \mathbf{L}_1,\ A_k \in \mathbf{M}_1$;
(c2) $0 < \rho \leq \tau_{j+1} - \tau_j \leq \sigma$;
(c3) $\| B_j \| \leq B < 1$;
(c4) there exists $\delta > 0$ *such that* $t - h_k(t) < \delta$;
(c5) $e^{\eta(\sigma+\delta+1)} \sum_{k=1}^{m} \| A_k \|_{\mathbf{M}_1} \leq 1 - e^{-\eta}$, *where* $\eta = -\frac{1}{\sigma} \ln B$,
hold.
Then the equation (1) - (3) is exponentially stable.

Proof of Theorem 5. First we note that the hypotheses of the theorem imply (a1)-(a6). In particular, (b2) implies (a1) and (a6). By Theorem 1 the hypotheses of the theorem ensure admissibility of the pair $(\mathbf{D}_1, \mathbf{L}_1)$ for operator $\mathcal{L}$ defined by (4).

The hypotheses of Theorem 2 are satisfied if operator $\mathcal{L}C_{\mathcal{M}} : \mathbf{L}_1 \ \to \mathbf{L}_1$ is invertible, where $C_{\mathcal{M}}$ is the Cauchy operator of the problem (22).

Evidently $\mathcal{L}C_{\mathcal{M}} = E + T$, where

$$(Tz)(t) = \sum_{k=1}^{m} A_k(t) \int_0^{h_k^+(t)} X_1(h_k(t), s) z(s) ds.$$

Lemma 6 gives that the operator $C_{\mathcal{M}}$ acts from $\mathbf{L}_1$ to $\mathbf{D}_1$. Since by the hypothesis of the theorem $A_k^\eta \in \mathbf{M}_1$, then from the equality $T = HC_{\mathcal{M}}$, where H is defined by (9), and from Theorem 1 the operator T acts in $\mathbf{L}_1$.

Let estimate the norm of operator T:

$$\| Tz \|_{\mathbf{L}_1} \leq \sum_{k=1}^{m} \int_0^\infty \| A_k(t) \| \int_0^{h_k^+(t)} e^{-\eta(h_k(t)-s-\sigma)} \| z(s) \| \, ds \, dt \leq$$

$$\leq e^{\eta\sigma} \sum_{k=1}^{m} \int_0^\infty \int_0^t \| A_k(t) e^{\eta(t-h_k(t))} e^{-\eta(t-s)} \| z(s) \| \, ds \, dt =$$

$$= e^{\eta\sigma} \sum_{k=1}^{m} \int_0^\infty \left(\int_s^\infty \| A_k^\eta(t) \| e^{-\eta t} dt \right) e^{\eta s} \| z(s) \| \, ds \leq$$

$$\leq e^{\eta\sigma} \sum_{k=1}^{m} \int_0^\infty \left(\sum_{i=[s]}^{\infty} \int_i^{i+1} \| A_k^\eta(t) \| e^{-\eta t} dt \right) e^{\eta s} \| z(s) \| \, ds \leq$$

$$\leq e^{\eta\sigma} \sum_{k=1}^{m} \| A_k^\eta \|_{\mathbf{M}_1} \int_0^\infty \sum_{i=[s]}^{\infty} e^{-\eta i} e^{\eta s} \| z(s) \| \, ds =$$

$$= e^{\eta\sigma} \sum_{k=1}^{m} \| A_k^\eta \|_{\mathbf{M}_1} \frac{e^\eta}{1-e^{-\eta}} \| z \|_{\mathbf{L}_1} .$$

The hypothesis (b5) implies $\| T \|_{\mathbf{L}_1 \to \mathbf{L}_1} < 1$, therefore $\mathcal{L}C_{\mathcal{M}} : \mathbf{L}_1 \to \mathbf{L}_1$ is invertible. Hence all hypotheses of Theorem 2 hold. The proof of the theorem is complete.

Proof of Theorem 6. The hypothesis (c4) implies $\varphi(h_k(t)) = 0$ for $t > \delta$. Thus (b1),(b4),(b5) and other hypotheses of Theorem 5 hold. By Theorem 4 the equation (1)-(3) is exponentially stable.

Example. Consider a scalar equation

$$\dot{x}(t) + a(t)x(\lambda t) = f(t), \; t \geq 0, \; 0 < \lambda < 1,$$
$$x(j) = bx(j-0), \; j = 1, 2, \dots, \; | b | < 1. \tag{24}$$

Since $h(t) = \lambda t \geq 0$ then one may assume $\varphi \equiv 0$.

The constant η defined in (b5) is $\eta = -\ln b$. Therefore by Theorem 5 all solutions of (24) are in $\mathbf{L}_1$ for any $f \in \mathbf{L}_1$, i.e. they are integrable on the half-line if

$$a^\eta(t) = a(t) e^{[(\lambda-1)\ln b]t} \in \mathbf{M}_1 \quad \text{and} \quad \| a^\eta \|_{\mathbf{M}_1} \leq (1-b)b^2.$$

7 Proofs of Lemmas 3 and 4

Lemma 3. *Suppose (a5) and (a6) hold. Then* $\mathbf{D}_p$, $1 \le p < \infty$, *is a Banach space.*

Proof. Let $\{x_j\}$ be a fundamental sequence in $\mathbf{D}_p$, i.e.

$$\lim_{k,i\to\infty} \| x_k - x_i \|_{\mathbf{D}_p} = 0.$$

First we will prove that $\{x_j(0)\}$ converges in $\mathbf{R}^n$.

The convergence $\| y_j \|_{\mathbf{D}_p[0,\infty)} \to 0$ implies $\| y_j \|_{\mathbf{D}_p[0,t_0]} \to 0$ for any $t_0 > 0$. Hence $\| y_j \|_{\mathbf{L}_1[0,t_0]} \to 0$, $\| \dot{y}_j \|_{\mathbf{L}_1[0,t_0]} \to 0$. Therefore for $t_0 < \tau_1$ and $y_j = x_k - x_i$ we have

$$\lim_{k,i\to\infty} \| x_k - x_i \|_{\mathbf{L}_1[0,t_0]} = 0, \lim_{k,i\to\infty} \| \dot{x}_k - \dot{x}_i \|_{\mathbf{L}_1[0,t_0]} = 0.$$

Consider an identity

$$x_k(t) - x_i(t) = x_k(0) - x_i(0) + \int_0^t [\dot{x}_k(s) - \dot{x}_i(s)] ds.$$

Since

$$\lim_{k,i\to\infty} \| x_k - x_i \|_{\mathbf{L}_1[0,t_0]} = 0 \text{ and}$$

$$\lim_{k,i\to 0} \int_0^{t_0} \int_0^t | \dot{x}_k(s) - \dot{x}_i(s) | \ ds\ dt \le t_0 \lim_{k,i\to\infty} \| \dot{x}_k - \dot{x}_i \|_{\mathbf{L}_1[0,t_0]} = 0,$$

then

$$\lim_{k,i\to\infty} \| x_k(0) - x_i(0) \|_{\mathbf{L}_1[0,t_0]} = 0.$$

Hence

$$\lim_{k,i\to\infty} \| x_k(0) - x_i(0) \|_{\mathbf{L}_1[0,t_0]} = t_0 \lim_{k,i\to\infty} \| x_k(0) - x_i(0) \|_{\mathbf{R}^n} = 0,$$

i.e. the sequence $\{x_j(0)\}$ is fundamental in $\mathbf{R}^n$. Therefore there exists $\beta \in \mathbf{R}^n$ such that $\lim_{j\to\infty} x_j(0) = \beta$.

Let $f_j = \mathcal{L}_0 x_j$, where operator $\mathcal{L}_0$ is defined by (6), $\nu = a - I \ln b > 0$. Then by Lemma 1

$$x_j(t) = X_0(t,0)x_j(0) + \int_0^t X_0(t,s) f_j(s) ds. \tag{25}$$

Lemma 2 yields

$$\| X_0(t,0) \| \leq e^{-\nu t}.$$

Since $\dot{X}_0(t,0) + aX_0(t,0) = 0$ then

$$\| \dot{X}_0(t,0) \| \leq ae^{-\nu t}.$$

Therefore the sequence $\{X_0(t,0)x_j(0)\}$ converges in $\mathbf{D}_p$ to the function $X_0(t,0)\beta$.

By Lemma 2 we obtain that the operators $\mathcal{L}_0 : \mathbf{D}_p \to \mathbf{L}_p$ and $C_0 : \mathbf{L}_p \to \mathbf{D}_p$ are bounded. To this end denoting $x = C_0 f$ we obtain

$$\| \mathcal{L}_0 x \|_{\mathbf{L}_p} \leq \| \dot{x} \|_{\mathbf{L}_p} + a \| x \|_{\mathbf{L}_p} \leq (1+a) \| x \|_{\mathbf{D}_p} .$$

By Lemma 2 operator C_0 is bounded in $\mathbf{L}_p$ [1], hence

$$\| C_0 f \|_{\mathbf{D}_p} = \| x \|_{\mathbf{L}_p} + \| \dot{x} \|_{\mathbf{L}_p} \leq \| C_0 \|_{\mathbf{L}_p \to \mathbf{L}_p} \| f \|_{\mathbf{L}_p} + \| f - ax \|_{\mathbf{L}_p} \leq$$

$$\leq [1 + \| C_0 \| (1+a)] \| f \|_{\mathbf{L}_p} .$$

Since $\mathcal{L}_0 : \mathbf{D}_p \to \mathbf{L}_p$ is continuous and $\mathcal{L}_0 x_j = f_j$ then $\{f_j\}$ is a fundamental sequence. Therefore there exists $f \in \mathbf{L}_p$ such that $\lim_{j\to\infty} f_j = f$.

Let $\tilde{x} = C_0 f,\ \tilde{x}_j = C_0 f_j$. The continuity of the operator $C_0 : \mathbf{L}_p \to \mathbf{D}_p$ implies $\| \tilde{x}_j - \tilde{x} \|_{\mathbf{D}_p} \to 0$.

From here sequence

$$x_j(t) = X_0(t,0)x_j(0) + \tilde{x}_j(t)$$

converges in $\mathbf{D}_p$ to

$$x(t) = X_0(t,0)\beta + \tilde{x}(t).$$

The proof of the lemma is complete.

<u>Lemma 4.</u> *Let* $a > I \ln b$.
Then the set

$$\tilde{\mathbf{D}_p} = \{x \in \mathbf{PAC} \mid \dot{x} + ax \in \mathbf{L}_p,\ x(\tau_j) = B_j x(\tau_j - 0)\}$$

coincides with $\mathbf{D}_p$. *Besides the norm*

$$\| x \|_{\tilde{\mathbf{D}_p}} = \| x(0) \| + \| \dot{x} + ax \|_{\mathbf{L}_p} \tag{26}$$

is equivalent to the norm in $\mathbf{D}_p$.

Proof. Let $x \in \tilde{\mathbf{D}}_p$ and $z = \dot{x} + ax$. Then $x(t) = X_0(t,0)x(0) + (C_0 z)(t)$. By Lemma 2 $z \in \mathbf{L}_p$ implies $x \in \mathbf{L}_p$. Hence $\dot{x} = z - ax \in \mathbf{L}_p$, thus $x \in \mathbf{D}_p$.

Let $x \in \mathbf{D}_p$. Then the inequality

$$\| \dot{x} + ax \|_{\mathbf{L}_p} \leq (1+a) \| x \|_{\mathbf{D}_p}$$

implies $\dot{x} + ax \in \mathbf{L}_p$. Hence $x \in \tilde{\mathbf{D}}_p$. Thus $\tilde{\mathbf{D}}_p = \mathbf{D}_p$.

Formula (26) defines a norm in $\mathbf{D}_p$. In fact if $\| x \|_{\tilde{\mathbf{D}}_p} = 0$ then $\dot{x} + ax = 0,\ x(0) = 0$. Then by Lemma 1 on uniqueness of a solution $x = 0$.

Let us prove that the space $\mathbf{D}_p$ endowed with the norm $\| \cdot \|_{\tilde{\mathbf{D}}_p}$ is complete. Suppose $\{x_j\}$ is a fundamental sequence by this norm. Denote $y_j = \dot{x}_j + ax_j$. Then the convergence

$$\| x_k(0) - x_i(0) \| + \| y_k - y_i \|_{\mathbf{L}_p} \to 0 \ \text{ for } \ k, i \to \infty$$

implies $\{x_j(0)\}$ is fundamental in $\mathbf{R}^n$ and $\{y_j\}$ is fundamental in $\mathbf{L}_p$. Therefore these sequences converge in the corresponding spaces.

Consider the equality

$$x_j(t) = X_0(t,0)x_j(0) + (C_0 y_j)(t). \tag{27}$$

We will prove that the operator $C_0 : \mathbf{L}_p \to \tilde{\mathbf{D}}_p$ is bounded. Let $x = C_0 f$. Then $x(0) = 0$ and

$$\| C_0 f \|_{\tilde{\mathbf{D}}_p} = \| \dot{x} + ax \|_{\mathbf{L}_p} = \| f \|_{\mathbf{L}_p} .$$

Boundedness of $C_0 : \mathbf{L}_p \to \tilde{\mathbf{D}}_p$ and the equality (27) yield the convergence of $\{x_j\}$ in $\tilde{\mathbf{D}}_p$. Consequently this space is complete.

Consider sets

$$\mathbf{D}_p^0 = \{x \in \mathbf{D}_p \mid x(0) = 0\},$$
$$U_n = \{x = X_0(t,0)\alpha \mid \ \alpha \in \mathbf{R}^n\}.$$

The space U_n is n-dimensional, isomorphic to $\mathbf{R}^n$ and $U_n \subset \mathbf{D}_p$. Since

$$x(t) = X(t,0)x(0) + \int_0^t X_0(t,s)[\dot{x}(s) + ax(s)]ds,$$

then $\mathbf{D}_p$ is aljebraically isomorphic to the direct sum $\mathbf{D}_p^0 \oplus U_n$.

Since U_n is finite-dimensional then [15] the subspace $\mathbf{D}_p^0$ is closed in $\mathbf{D}_p$ and in $\tilde{\mathbf{D}}_p$.

First we will prove equivalence of norms $\|\cdot\|_{\mathbf{D}_p}$ and $\|\cdot\|_{\tilde{\mathbf{D}}_p}$ in $\mathbf{D}_p^0$. Let $x \in \mathbf{D}_p^0$. To this end

$$\| x \|_{\tilde{\mathbf{D}}_p} = \| \dot{x} + ax \|_{\mathbf{L}_p} \leq (1+a) \| x \|_{\mathbf{D}_p} .$$

From here and from the fact D_p^0 is a Banach space with both norms we obtain [15] that in D_p^0 these norms are equivalent.

Let P_1 and P_2 be projectors to subspaces $\mathbf{D}_p^0$ and U_n correspondingly. $\mathbf{D}_p^0$ is closed, therefore these projectors are bounded operators in $\mathbf{D}_p$ and $\tilde{\mathbf{D}}_p$. Let $\| x_j \|_{\mathbf{D}_p} \to 0$. Then the relations

$$x_j = P_1 x_j + P_2 x_j, \quad \| P_i x_j \|_{\mathbf{D}_p} \leq \| P_i \| \| x_j \|_{\mathbf{D}_p}, \ i = 1, 2,$$

imply $\| P_i x_j \|_{\mathbf{D}_p} \to 0, \ i = 1, 2$. As $P_1 x_j \in \mathbf{D}_p^0$, and in $\mathbf{D}_p^0$ the norms $\|\cdot\|_{\mathbf{D}_p}$ and $\|\cdot\|_{\tilde{\mathbf{D}}_p}$ are equivalent, then $\| P_1 x_j \|_{\tilde{\mathbf{D}}_p} \to 0$.

Besides this $P_2 x_j \in U_n$. The space U_n is finite-dimensional and all the norms in it are equivalent. Thus $\| P_2 x_j \|_{\tilde{\mathbf{D}}_p} \to 0$. Consequently,

$$\| x_j \|_{\tilde{\mathbf{D}}_p} \leq \| P_1 x_j \|_{\tilde{\mathbf{D}}_p} + \| P_2 x_j \|_{\tilde{\mathbf{D}}_p} \to 0.$$

Therefore the norms $\|\cdot\|_{\mathbf{D}_p}$ and $\|\cdot\|_{\tilde{\mathbf{D}}_p}$ are equivalent, which completes the proof.

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